PATTERN AVOIDANCE IN GENERALIZED NON-CROSSING TREES

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Abstract. In this paper, the problem of pattern avoidance in generalized non-crossing trees is studied. The generating functions for generalized non-crossing trees avoiding patterns of length one and two are obtained. Lagrange inversion formula is used to obtain the explicit formulas for some special cases. Bijection is also established between generalized non-crossing trees with special pattern avoidance and the little Schröder paths.

Keywords: Generalized non-crossing tree, pattern avoidance, Catalan number, little Schröder path.

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1. Introduction

A *non-crossing tree* (NC-tree for short) is a tree drawn on n points in $\{1, 2, \cdots, n\}$ numbered in counterclockwise order on a circle such that the edges lie entirely within the circle and do not cross. Non-crossing trees have been investigated by Chen and Yan [1], Deutsch and Noy [3], Flajolet and Noy [4], Gu, et al. [5], Hough [6], Noy [8], Panholzer and Prodinger [9]. Recently, some problems of pattern avoidance in NC-trees have been studied by Sun and Wang [13]. It is well known that the set of NC-trees with $n+1$ vertices is counted by the generalized Catalan number $\frac{1}{2n+1}\binom{3n}{n}$ [11, A001764].

A *generalized non-crossing tree* (GNC-tree for short) is a modified NC-tree such that the labels are weakly increasing in counterclockwise order and if $j \geq 1$ is a label then all $1 \leq i \leq j$ are also labels. See Figure 2 for example.

In the sequel, we are concerned with the rooted GNC-trees such that the first 1 is the root. Let GNC_n denote the set of rooted GNC-trees of $n+1$ vertices. It is easy to prove that GNC_n is counted by $|\mathsf{GNC}_n| = \frac{2^n}{2n+1}\binom{3n}{n}$ [11, not listed], for a bar can be or not be inserted into any position between i and $i+1$ for $1 \leq i \leq n$ in an NC-tree of $n+1$ points and assume that one bar always appears in the position between $n+1$ and 1, then relabel the numbers between any two bars in a proper way to form a GNC-tree.

A *descent (an ascent, a level)* is an edge (i, j) such that $i > j$ $(i < j, i = j)$ and i is on the path from the root to the vertex j. If encoding an ascent by u, a level by h and a descent by d, then each path in a GNC-tree can be represented by a ternary word on $\{u, h, d\}$ by viewing from the root. In analogy with the well-established permutation patterns [10, 14], we propose a definition of patterns in GNC-trees.

Definition 1.1. *Let $w = w_1w_2 \ldots w_n$ and $\sigma = \sigma_1\sigma_2 \ldots \sigma_k$ be two ternary words on $\{u, h, d\}$. Then w contains the pattern σ if it has a subword $w_{i+1}w_{i+2} \ldots w_{i+k}$ equal to σ for some $0 \leq i \leq n-k$; otherwise w is called σ-avoiding. A* GNC*-tree T is called σ-avoiding if T has no subpath (viewing from the root) encoded by σ.*

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Let $\mathcal{P}_k$ denote the set of ternary words of length k on $\{u, h, d\}$. For any $\sigma \in \mathcal{P}_k$, let $\mathsf{GNC}_n^m(\sigma)$ denote the set of GNC-trees in GNC_n which contain the pattern σ exactly m times. For any nonempty subset $P \subset \mathcal{P}_k$, $\mathsf{GNC}_n(P)$ denotes the set of GNC-trees in GNC_n which avoid all the patterns in P. Analogous to restricted permutations, a counterpart in GNC-trees is the following question

Question 1.2. *Determine the cardinalities of $\mathsf{GNC}_n(P)$ for $P \subset \mathcal{P}_k$ and $\mathsf{GNC}_n^m(\sigma)$ for $\sigma \in \mathcal{P}_k$.*

In the literature, two kind of special NC-trees have been considered, that is non-crossing increasing trees and non-crossing alternating trees. Both of them are counted by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ [11, A000108]. *A non-crossing increasing (alternating) tree* is an NC-tree with the vertices on the path from the root 1 to any other vertex appearing in increasing (alternating) order. By our notation, a non-crossing increasing tree is just a d-avoiding NC-tree and a non-crossing alternating tree is just a $\{uu, dd\}$-avoiding NC-tree. Bijections between non-crossing alternating trees and Dyck paths have been presented in [12]. But for GNC-trees, it seems to be thrown little light on this subject.

In this paper, we deal with several patterns and find the corresponding generating functions for GNC-trees. More precisely, we investigate the patterns in $\mathcal{P}_1$ in Section 2 and the patterns in $\mathcal{P}_2$ in Section 3. Lagrange inversion formula is used to obtain the explicit formulas for some special cases. Bijection is also established between GNC-trees with special pattern avoidance and the little Schröder paths.

2. The patterns in $\mathcal{P}_1$

For any $T \in \mathsf{GNC}_n$, let $u(T), h(T), d(T)$ denote the number of ascents, levels and descents of T respectively, then $u(T) + h(T) + d(T) = n$. Let GNC_n^* be the set of GNC-trees T in GNC_n such that T has only one point with label 1, namely only the root has the label 1 and others have labels greater than 1. Define

$$
\begin{aligned}
T_{x,y,z}(t) &= \sum_{n\geq 0} t^n \sum_{T\in \mathsf{GNC}_n} x^{u(T)} y^{h(T)} z^{d(T)},\\
T^*_{x,y,z}(t) &= \sum_{n\geq 0} t^n \sum_{T\in \mathsf{GNC}^*_n} x^{u(T)} y^{h(T)} z^{d(T)}.
\end{aligned}
$$

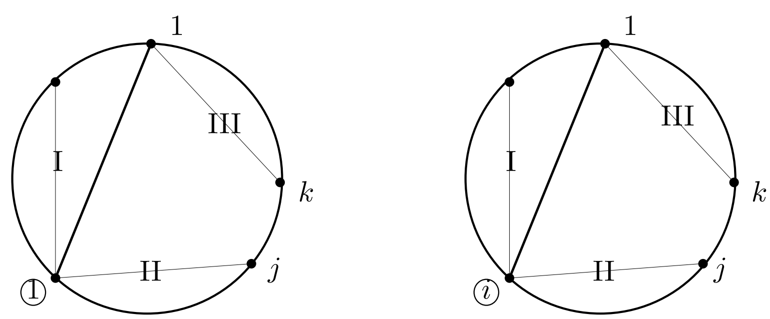

Figure 1. Decomposition of GNC-trees.

Close relations between $T_{x,y,z}(t)$ and $T^*_{x,y,z}(t)$ can be established according to the decomposition of GNC-trees in Figure 1. Find the first and minimal label i, denoted by i, of $T \in \mathsf{GNC}_n$ in counterclockwise order such that the root 1 and i form an edge, then T can be partitioned into three parts.

(i) The case $i = j = 1$ and $k = 1$ or 2. Part I and II both avoid the patterns u and d, Part III still forms a GNC-tree. Then the edge $(1, 1)$ contributes an yt, each of Part I and II contributes $T_{0,y,0}(t)$, and Part III contributes $T_{x,y,z}(t)$.

(ii) The case $i=1, j\geq 2$ and $k=j$ or $j+1$. Part I avoids the patterns u and d, Part II still forms a GNC-tree different from Part I. Then the edge $(1,1)$ contributes an yt, Part I contributes $T_{0,y,0}(t)$ and Part II contributes $T_{x,y,z}(t)-T_{0,y,0}(t)$. For Part III, except for the one point case, decreasing all the labels (excluding the root), by $j-2$ units (if $k=j$) or $j-1$ units (if $k=j+1$), one can obtain two GNC-trees in GNC_n^* for some $n\geq 1$. Then Part III contributes $2T^*_{x,y,z}(t)-1$.

(iii) The case $i\geq 2, j\geq i$ and $k=j$ or $j+1$. The edge $(1,i)$ contributes an xt. Note that the ascents and descents are exchanged in Part I, and the labels are lying in $\{1,2,\ldots,i\}$ or $\{2,3,\ldots,i\}$, so Part I contributes $2T_{z,y,x}(t)-T_{0,y,0}(t)$ for $i\geq 2$. Part II still forms a GNC-tree after reducing the labels $\{i,\ldots,j\}$ to $\{1,\cdots,j-i+1\}$, so Part II contributes $T_{x,y,z}(t)$. Similar to (ii), Part III contributes $2T^*_{x,y,z}(t)-1$.

Summarizing these, we have

$$\begin{aligned} T_{x,y,z}(t) &= 1+ytT_{0,y,0}(t)^2T_{x,y,z}(t)\\ &\quad + ytT_{0,y,0}(t)\Big\{T_{x,y,z}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^*_{x,y,z}(t)-1\Big\}\\ &\quad + xtT_{x,y,z}(t)\Big\{2T_{z,y,x}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^*_{x,y,z}(t)-1\Big\}. \end{aligned} \tag{2.1}$$

For any $T\in\mathsf{GNC}_n^*$, by the similar decomposition, one can derive that

$$T^*_{x,y,z}(t)=1+xtT_{x,y,z}(t)T_{z,y,x}(t)\Big\{2T^*_{x,y,z}(t)-1\Big\}. \tag{2.2}$$

Solve (2.2) for $T^*_{x,y,z}(t)$ and substitute it into (2.1), one can get

$$\begin{aligned} T_{x,y,z}(t) &= 1-ytT_{0,y,0}(t)^2-xtT_{x,y,z}(t)T_{0,y,0}(t)+ytT_{0,y,0}(t)\Big\{1+T_{0,y,0}(t)\Big\}T_{x,y,z}(t)\\ &\quad + 2xt\Big\{1-ytT_{0,y,0}(t)^2\Big\}T_{x,y,z}(t)^2T_{z,y,x}(t). \end{aligned} \tag{2.3}$$

Let $x=z=0$, (2.3) reduces to

$$T_{0,y,0}(t) = 1+ytT_{0,y,0}(t)^3. \tag{2.4}$$

Multiplying by $T_{0,y,0}(t)$ in both side of (2.3) and using (2.4), after some routine computations, one can deduce that

$$T_{x,y,z}(t) = 1+(y-x)tT_{0,y,0}(t)^2T_{x,y,z}(t)+2xtT_{x,y,z}(t)^2T_{z,y,x}(t). \tag{2.5}$$

Solve (2.5) for $T_{x,y,z}(t)$, we have

$$\begin{aligned} T_{x,y,z}(t) &= \frac{1-(y-x)tT_{0,y,0}(t)^2-\sqrt{(1-(y-x)tT_{0,y,0}(t)^2)^2-8xtT_{z,y,x}(t)}}{4xtT_{z,y,x}(t)}\\ &= \frac{1}{1-(y-x)tT_{0,y,0}(t)^2}C\Big(\frac{2xtT_{z,y,x}(t)}{(1-(y-x)tT_{0,y,0}(t)^2)^2}\Big), \end{aligned} \tag{2.6}$$

where $C(t)=\frac{1-\sqrt{1-4t}}{2t}$ is the generating function for Catalan numbers.

Exchanging x and z in (2.6), one can obtain $T_{z,y,x}(t)$, and then substitute it into (2.6), one has the following proposition.

Proposition 2.1. *The generating functions $T_{x,y,z}(t)$ for GNC-trees satisfies*

$$T_{x,y,z}(t) = \alpha C\big(2xt\alpha^2\beta C\big(2zt\beta^2T_{x,y,z}(t)\big)\big),$$

where $\alpha=\frac{1}{1-(y-x)tT_{0,y,0}(t)^2}$ and $\beta=\frac{1}{1-(y-z)tT_{0,y,0}(t)^2}$.

By Lagrange inversion formula and some series expansions, one can obtain the coefficients of t^n of $T_{x,y,z}(t)$, but it seems to be somewhat complicated. Now we will consider several special cases which lead to interesting results.

2.1. u**-avoiding GNC-trees.** Note that $T_{0,y,z}(t)$ is the generating function for GNC-trees with no ascent. Let $x=0$ in (2.5), one can find

$$T_{0,y,z}(t) \quad = \quad 1+ytT_{0,y,0}(t)^2T_{0,y,z}(t),$$

from which, together with (2.4), using Lagrange inversion formula, one can deduce that

$$T_{0,y,z}(t)=T_{0,y,0}(t)=\sum_{n\geq 0}\frac{1}{2n+1}\binom{3n}{n}(yt)^n=T_{0,1,0}(yt).$$

In fact, the above relation can be easily derived from the definition of GNC-tree, for a u-avoiding GNC-tree must also avoid the pattern d, such GNC-trees can be obtained by changing each label of the underlying NC-trees to the label 1.

2.2. h**-avoiding GNC-trees.** Note that $T_{x,0,z}(t)$ is the generating function for GNC-trees with no level. Let $y=0$ in (2.4) and (2.5), one can find

$$T_{x,0,z}(t) \quad = \quad 1-xtT_{x,0,z}(t)+2xtT_{x,0,z}(t)^2T_{z,0,x}(t),$$

which, when $x=z=1$, generates

$$T_{1,0,1}(t) \quad = \quad 1-tT_{1,0,1}(t)+2tT_{1,0,1}(t)^3.$$

Let $\lambda=T_{1,0,1}(t)-1$, then $\lambda=t(1+\lambda)(2(1+\lambda)^2-1)$, using Lagrange inversion formula [15], one can deduce for $n\geq 1$ that

$$\begin{aligned}
[t^n]T_{1,0,1}(t) &= [t^n]\lambda=\frac{1}{n}[\lambda^{n-1}](1+\lambda)^n(2(1+\lambda)^2-1)^n\\
&= \frac{1}{n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}2^i[\lambda^{n-1}](1+\lambda)^{n+2i}\\
&= \sum_{i=0}^{n}(-1)^{n-i}\frac{2^i}{2i+1}\binom{n}{i}\binom{n+2i}{n}\\
&= \sum_{i=0}^{n}(-1)^{n-i}\frac{2^i}{2i+1}\binom{3i}{i}\binom{n+2i}{3i}.
\end{aligned}$$

Hence we have

Theorem 2.2. *The set of h-avoiding GNC-trees of $n+1$ points is counted by*

$$|\mathsf{GNC}_n(h)| \quad = \quad \sum_{i=0}^{n}(-1)^{n-i}\frac{2^i}{2i+1}\binom{3i}{i}\binom{n+2i}{3i}.$$

This sequence beginning with $1,1,5,31,217,1637,12985$ is not listed in Sloane's [11].

2.3. d**-avoiding GNC-trees.** Note that $T_{x,y,0}(t)$ is the generating function for GNC-trees with no descent. Let $z=0$ in (2.6), we can find

$$\begin{aligned}
T_{x,y,0}(t) &= \frac{1}{1-(y-x)tT_{0,y,0}(t)^2}C\Big(\frac{2xtT_{0,y,x}(t)}{(1-(y-x)tT_{0,y,0}(t)^2)^2}\Big),\\
&= \frac{1}{1-(y-x)tT_{0,1,0}(yt)^2}C\Big(\frac{2xtT_{0,1,0}(yt)}{(1-(y-x)tT_{0,1,0}(yt)^2)^2}\Big), \qquad (2.7)
\end{aligned}$$

where we use the relation $T_{0,y,x}(t)=T_{0,y,0}(t)=T_{0,1,0}(yt)$.

Case i. Let $x = y = 1$ in (2.7), one gets

$$T_{1,1,0}(t) \quad = \quad C(2tT_{0,1,0}(t)).$$

Taking the coefficient t^n of $T_{1,1,0}(t)$, one gets

$$\begin{aligned}
[t^n]T_{1,1,0}(t) &= [t^n]C(2tT_{0,1,0}(t)) = [t^n]\sum_{i\geq 0} 2^i C_i t^i T_{0,1,0}(t)^i \\
&= [t^n]\sum_{i\geq 0} 2^i C_i t^i \sum_{j\geq 0} \frac{i}{3j+i}\binom{3j+i}{j} t^j, \\
&= \sum_{i+j=n} \frac{i}{3j+i}\binom{3j+i}{j} 2^i C_i.
\end{aligned}$$

Hence we have

Theorem 2.3. *The set of d-avoiding GNC-trees of $n+1$ points is counted by*

$$|\mathsf{GNC}_n(d)| \quad = \quad \sum_{i+j=n} \frac{i}{3j+i}\binom{3j+i}{j} 2^i C_i.$$

This sequence beginning with $1, 2, 10, 62, 424, 3070$ is not listed in Sloane's [11].

Case ii. Let $y = 1$ in (2.7), by (2.4), one gets

$$T_{x,1,0}(t) \quad = \quad \frac{T_{0,1,0}(t)}{1 + xtT_{0,1,0}(t)^3} C\left(\frac{2xtT_{0,1,0}(t)^3}{(1 + xtT_{0,1,0}(t)^3)^2}\right).$$

Taking the coefficient $t^n x^k$ of $T_{x,1,0}(t)$, one gets

$$\begin{aligned}
[t^n x^k]T_{x,1,0}(t) &= [t^{n-k}x^k]\frac{T_{0,1,0}(t)}{1 + xT_{0,1,0}(t)^3} C\left(\frac{2xT_{0,1,0}(t)^3}{(1 + xT_{0,1,0}(t)^3)^2}\right) \\
&= [t^{n-k}]\sum_{i=0}^{k} 2^i C_i T_{0,1,0}(t)^{3i+1} [x^{k-i}] \frac{1}{(1 + xT_{0,1,0}(t))^{2i+1}} \\
&= \sum_{i=0}^{k} 2^i C_i (-1)^{k-i}\binom{k+i}{k-i} [t^{n-k}]T_{0,1,0}(t)^{k+2i+1} \\
&= \sum_{i=0}^{k} (-1)^{k-i}\binom{k+i}{k-i} \frac{k+2i+1}{3n-2k+2i+1}\binom{3n-2k+2i+1}{n-k} 2^i C_i.
\end{aligned}$$

Hence we have

Theorem 2.4. *The set of d-avoiding GNC-trees of $n+1$ points with k ascents is counted by*

$$\sum_{i=0}^{k} (-1)^{k-i}\binom{k+i}{k-i} \frac{k+2i+1}{3n-2k+2i+1}\binom{3n-2k+2i+1}{n-k} 2^i C_i.$$

Case iii. Let $x = 1, y = 0$ in (2.7), using $T_{0,0,0}(t) = T_{0,1,0}(0) = 1$, one gets

$$T_{1,0,0}(t) \quad = \quad \frac{1}{1+t} C\left(\frac{2t}{(1+t)^2}\right) = \frac{1 + t - \sqrt{1 - 6t + t^2}}{4t},$$

which is the generating function for little Schröder paths. A *little Schröder path of length* $2n$ is a lattice path in the first quadrant going from $(0,0)$ to $(2n,0)$ consisting of up steps $U = (1,1)$, down steps $D = (1,-1)$ and horizontal steps $H_\ell H_r = (2,0)$ with no horizontal step at the x-axis. Let $\mathscr{R}_n$ denote the set of little Schröder paths of length $2n$ which is

counted by the nth little Schröder number R_n [11, A001003], whose generating function is $R(t) = \frac{1+t-\sqrt{1-6t+t^2}}{4t}$. Hence we have

Theorem 2.5. *The set of $\{h, d\}$-avoiding (i.e. increasing) GNC-trees of $n+1$ points is counted by the nth little Schröder numbers. In other words, there exists a bijection between $\mathsf{GNC}_n(h, d)$ and $\mathscr{R}_n$.*

Proof. Read any $\{h, d\}$-avoiding GNC-tree of $n+1$ points in preorder, denote an ascent (i, j) by U if it is read in the first time and not following another ascent (i, j); Denote an ascent (i, j) by H_ℓ if it is read in the second time and followed by another ascent (i, j) which is then denoted by H_r; Denote an ascent (i, j) by D if it is read in the second time and not followed by another ascent (i, j). Then we can get a little Schröder of length $2n$. The above procedure is clearly invertible, see Figure 2. □

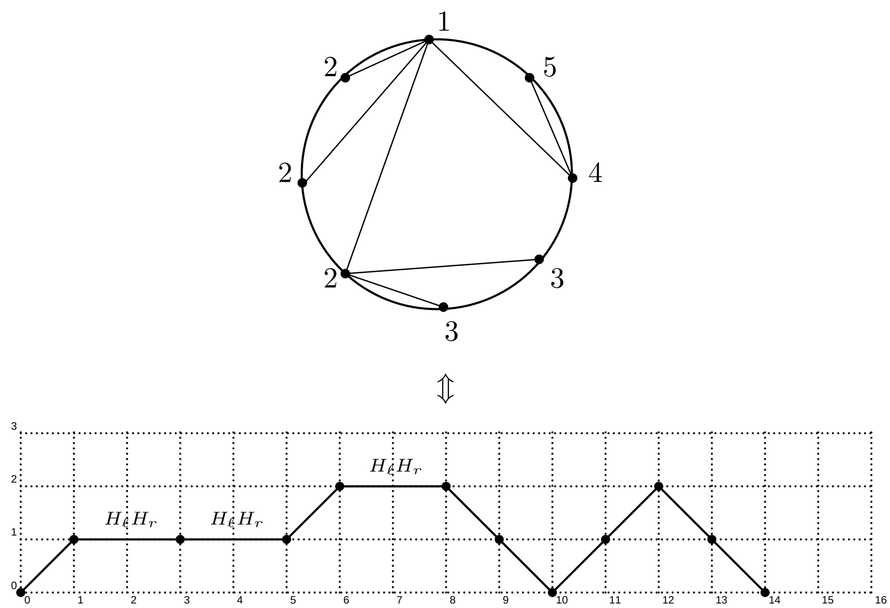

FIGURE 2. The bijection between $\mathsf{GNC}_n(h, d)$ and $\mathscr{R}_n$.

3. The patterns in $\mathcal{P}_2$

Let $\mathsf{GNC}^*_n(\sigma)$ be the set of σ-avoiding GNC-trees T in GNC^*_n with $\sigma \in \mathcal{P}_2$. Define

$$T^{\sigma}_{x,y,z}(t) = \sum_{n\geq 0} t^n \sum_{T\in \mathsf{GNC}_n(\sigma)} x^{u(T)} y^{h(T)} z^{d(T)},$$

$$T^{*\sigma}_{x,y,z}(t) = \sum_{n\geq 0} t^n \sum_{T\in \mathsf{GNC}^*_n(\sigma)} x^{u(T)} y^{h(T)} z^{d(T)}.$$

In this section, we will deal with the patterns uu, dd, ud and du, the others can be investigated similarly.

3.1. **The patterns uu and dd.** Close relations between $T^{uu}_{x,y,z}(t)$ and $T^{*uu}_{x,y,z}(t)$ can be established according to the decomposition of GNC-trees in Figure 1.

(i) The case $i = j = 1$ and $k = 1$ or 2. Part I and II both avoid the patterns u and d, Part III still forms a uu-avoiding GNC-tree. Then the edge $(1, 1)$ contributes an yt, each of Part I and II contributes $T_{0,y,0}(t)$, and Part III contributes $T^{uu}_{x,y,z}(t)$.

(ii) The case $i=1, j\geq 2$ and $k=j$ or $j+1$. Part I avoids the patterns u and d, Part II still forms a uu-avoiding GNC-tree different from Part I. Then the edge $(1,1)$ contributes an yt, Part I contributes $T_{0,y,0}(t)$ and Part II contributes $T^{uu}_{x,y,z}(t)-T_{0,y,0}(t)$. For Part III, except for the one point case, decreasing all the labels (excluding the root), by $j-2$ units if $k=j$ or $j-1$ units if $k=j+1$, one can obtain two uu-avoiding GNC-trees in $\mathsf{GNC}^*_n(uu)$ for some $n\geq 1$. Then Part III contributes $2T^{*uu}_{x,y,z}(t)-1$.

(iii) The case $i\geq 2, j\geq i$ and $k=j$ or $j+1$. The edge $(1,i)$ contributes an xt. Note that the ascents and descents are exchanged in Part I, and the labels are lying in $\{1,2,\ldots,i\}$ or $\{2,3,\ldots,i\}$, so Part I contributes $2T^{dd}_{z,y,x}(t)-T_{0,y,0}(t)$. But Part II can not begin with a u edge, i.e., all edges (if exist) starting form i are h edges, so we should further partition Part II into three parts, by finding the last i, denoted by i^*, in counterclockwise order such that (i,i^*) is an h edge which contributes an yt, see Figure 3. Clearly, Part II_1 and II_2 both contribute $T_{0,y,0}(t)$, and Part II_3 contributes $T^{uu}_{x,y,z}(t)$. Similar to (ii), Part III contributes $2T^{*uu}_{x,y,z}(t)-1$.

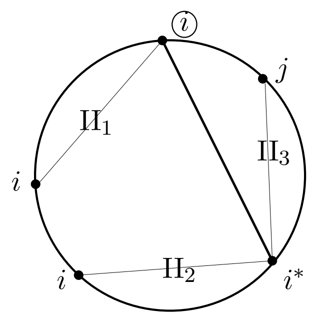

FIGURE 3. The decomposition of Part II in (iii).

Summarizing these, we have

$$\begin{aligned} T^{uu}_{x,y,z}(t) &= 1+ytT_{0,y,0}(t)^2T^{uu}_{x,y,z}(t)+\ ytT_{0,y,0}(t)\Big\{T^{uu}_{x,y,z}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^{*uu}_{x,y,z}(t)-1\Big\} \\ &\quad +xt\Big\{2T^{dd}_{z,y,x}(t)-T_{0,y,0}(t)\Big\}\Big\{1+ytT_{0,y,0}(t)^2T^{uu}_{x,y,z}(t)\Big\}\Big\{2T^{*uu}_{x,y,z}(t)-1\Big\}. \end{aligned} \tag{3.1}$$

For any $T\in\mathsf{GNC}^*_n(uu)$, by the similar decomposition, one can easily derive that

$$T^{*uu}_{x,y,z}(t)=1+xtT^{dd}_{z,y,x}(t)\Big\{1+ytT_{0,y,0}(t)^2T^{uu}_{x,y,z}(t)\Big\}\Big\{2T^{*}_{x,y,z}(t)-1\Big\}. \tag{3.2}$$

Similarly, for the pattern dd, close relations between $T^{dd}_{x,y,z}(t)$ and $T^{*dd}_{x,y,z}(t)$ can be established according to the decomposition of GNC-trees in Figure 1, the details are omitted.

$$\begin{aligned} T^{dd}_{x,y,z}(t) &= 1+ytT_{0,y,0}(t)^2T^{dd}_{x,y,z}(t)+ytT_{0,y,0}(t)\Big\{T^{dd}_{x,y,z}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^{*dd}_{x,y,z}(t)-1\Big\} \\ &\quad +xt\Big\{2T^{uu}_{z,y,x}(t)-T_{0,y,0}(t)\Big\}T^{dd}_{x,y,z}(t)\Big\{2T^{*dd}_{x,y,z}(t)-1\Big\}, \end{aligned} \tag{3.3}$$

$$T^{*dd}_{x,y,z}(t) = 1+xtT^{uu}_{z,y,x}(t)T^{dd}_{x,y,z}(t)\Big\{2T^{*dd}_{x,y,z}(t)-1\Big\}. \tag{3.4}$$

Solve (3.2) for $T^{*uu}_{x,y,z}(t)$ and (3.4) for $T^{*dd}_{x,y,z}(t)$, and then substitute them respectively into (3.1) and (3.3), after some simplifications, one can get

$$T^{uu}_{x,y,z}(t) = \Big\{1-xtT_{0,y,0}(t)^2+2xtT^{uu}_{x,y,z}(t)T^{dd}_{z,y,x}(t)\Big\}\Big\{1+ytT_{0,y,0}(t)^2T^{uu}_{x,y,z}(t)\Big\}, \tag{3.5}$$

$$T^{dd}_{x,y,z}(t) = 1+(y-x)tT_{0,y,0}(t)^2T^{dd}_{x,y,z}(t)+2xtT^{dd}_{x,y,z}(t)^2T^{uu}_{z,y,x}(t). \tag{3.6}$$

Let $x=y=z=1$ in (3.5) and (3.6), we have

Proposition 3.1. *The generating functions for uu-avoiding and dd-avoiding GNC-trees are determined respectively by*

$$\begin{aligned} T^{uu}_{1,1,1}(t) &= \Big\{1 - tT_{0,1,0}(t)^2 + 2tT^{uu}_{1,1,1}(t)T^{dd}_{1,1,1}(t)\Big\}\Big\{1 + tT_{0,1,0}(t)^2T^{uu}_{1,1,1}(t)\Big\}, \\ T^{dd}_{1,1,1}(t) &= 1 + 2tT^{dd}_{1,1,1}(t)^2T^{uu}_{1,1,1}(t). \end{aligned}$$

When $x = z = 1, y = 0$, by $T_{0,0,0}(t) = 1$, (3.5) and (3.6) generate

$$\begin{aligned} T^{uu}_{1,0,1}(t) &= 1 - t + 2tT^{uu}_{1,0,1}(t)T^{dd}_{1,0,1}(t), \\ T^{dd}_{1,0,1}(t) &= 1 - tT^{dd}_{1,0,1}(t) + 2tT^{dd}_{1,0,1}(t)^2T^{uu}_{1,0,1}(t), \end{aligned}$$

from which, one can deduce that

$$T^{uu}_{1,0,1}(t) = \frac{1-t}{1 - 2tT^{dd}_{1,0,1}(t)}, \tag{3.7}$$

$$T^{dd}_{1,0,1}(t) = 1 - 3tT^{dd}_{1,0,1}(t) + 4tT^{dd}_{1,0,1}(t)^2. \tag{3.8}$$

From (3.7) and (3.8), one can get

$$\begin{aligned} T^{dd}_{1,0,1}(t) &= \frac{1 + 3t - \sqrt{(1+3t)^2 - 16t}}{8t} \\ &= \frac{1}{1+3t}C\Big(\frac{4t}{(1+3t)^2}\Big) = \sum_{i\geq 0}\frac{4^iC_it^i}{(1+3t)^{2i+1}} \\ &= \sum_{i\geq 0}4^iC_it^i\sum_{j\geq 0}(-1)^j\binom{2i+j}{j}3^jt^j \\ &= \sum_{n\geq 0}t^n\sum_{j=0}^{n}(-1)^j\binom{2n-j}{j}3^j4^{n-j}C_{n-j}. \\ T^{uu}_{1,0,1}(t) &= \frac{1-t}{1-2tT^{dd}_{1,0,1}(t)} = \frac{3 - 3t - \sqrt{(1+3t)^2 - 16t}}{2} \\ &= \frac{3(1-t) - \sqrt{(1-t)^2 - 8t(1-t)}}{2} \\ &= 1 - t + 2tC\Big(\frac{2t}{1-t}\Big) = 1 + t + \sum_{i\geq 0}\frac{2^{i+2}C_{i+1}t^{i+2}}{(1-t)^{i+1}} \\ &= 1 + t + \sum_{n\geq 0}t^{n+2}\sum_{i=0}^{n}\binom{n}{i}2^{i+2}C_{i+1}. \end{aligned}$$

Hence we obtain

Theorem 3.2. *The sets $\mathsf{GNC}_{n+2}(uu,h)$ of $\{uu,h\}$-avoiding GNC-trees and $\mathsf{GNC}_n(dd,h)$ of $\{dd,h\}$-avoiding GNC-trees are counted respectively by*

$$|\mathsf{GNC}_{n+2}(uu,h)| = \sum_{i=0}^{n}\binom{n}{i}2^{i+2}C_{i+1}, \qquad \text{([11, not listed])},$$

$$|\mathsf{GNC}_{n}(dd,h)| = \sum_{j=0}^{n}(-1)^j\binom{2n-j}{j}3^j4^{n-j}C_{n-j}, \qquad \text{([11, A059231])}.$$

Remark 3.3. *Coker* [2] *proved that $T^{dd}_{1,0,1}(t)$ is also the generating function for $\mathscr{D}_n$, the set of different lattice paths running from $(0,0)$ to $(2n,0)$ using steps from $S = \{(k,\pm k)$:*

k positive integer$\}$ that never go below x-axis, and provided several different expressions for $|\mathscr{D}_n|$. One can be asked to find a bijection between $\mathscr{D}_n$ and $\mathsf{GNC}_n(dd,h)$.

3.2. **The patterns ud and du.** Similar to Subsection 3.1, close relations between $T^{ud}_{x,y,z}(t)$ and $T^{*ud}_{x,y,z}(t)$, and between $T^{du}_{x,y,z}(t)$ and $T^{*du}_{x,y,z}(t)$, can be derived according to the decomposition of GNC-trees, see in Figure 1, but the details are omitted.

$$\begin{aligned}
T^{ud}_{x,y,z}(t) &= 1+ytT_{0,y,0}(t)^2T^{ud}_{x,y,z}(t)+ytT_{0,y,0}(t)\Big\{T^{ud}_{x,y,z}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^{*ud}_{x,y,z}(t)-1\Big\}\\
&\quad+xtT^{ud}_{x,y,z}(t)\Big\{1+ytT_{0,y,0}(t)^2\{2T^{du}_{z,y,x}(t)-T_{0,y,0}(t)\}\Big\}\Big\{2T^{*ud}_{x,y,z}(t)-1\Big\}, \qquad (3.9)\\
T^{*ud}_{x,y,z}(t) &= 1+xtT^{ud}_{x,y,z}(t)\Big\{1+ytT_{0,y,0}(t)^2T^{du}_{z,y,x}(t)\Big\}\Big\{2T^{*ud}_{x,y,z}(t)-1\Big\}, \qquad (3.10)\\
T^{du}_{x,y,z}(t) &= 1+ytT_{0,y,0}(t)^2T^{du}_{x,y,z}(t)+ytT_{0,y,0}(t)\Big\{T^{du}_{x,y,z}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^{*du}_{x,y,z}(t)-1\Big\}\\
&\quad+xtT^{du}_{x,y,z}(t)\Big\{2T^{ud}_{z,y,x}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^{*du}_{x,y,z}(t)-1\Big\}, \qquad (3.11)\\
T^{*du}_{x,y,z}(t) &= 1+xtT^{du}_{x,y,z}(t)T^{ud}_{z,y,x}(t)\Big\{2T^{*du}_{x,y,z}(t)-1\Big\}. \qquad (3.12)
\end{aligned}$$

Solve (3.10) for $T^{*ud}_{x,y,z}(t)$ and (3.12) for $T^{*du}_{x,y,z}(t)$, and then substitute them respectively into (3.9) and (3.11), after some simplifications, one can get

$$\begin{aligned}
T^{ud}_{x,y,z}(t) &= 1+(y-x)tT_{0,y,0}(t)^2T^{ud}_{x,y,z}(t)+2xtT^{ud}_{x,y,z}(t)^2\Big\{1+ytT_{0,y,0}(t)^2T^{du}_{z,y,x}(t)\Big\}, \qquad (3.13)\\
T^{du}_{x,y,z}(t) &= 1+(y-x)tT_{0,y,0}(t)^2T^{du}_{x,y,z}(t)+2xtT^{du}_{x,y,z}(t)^2T^{ud}_{z,y,x}(t). \qquad (3.14)
\end{aligned}$$

Let $x=y=z=1$ in (3.13) and (3.14), we have

Proposition 3.4. *The generating functions for ud-avoiding and du-avoiding GNC-trees are given by*

$$\begin{aligned}
T^{ud}_{1,1,1}(t) &= 1+2tT^{ud}_{1,1,1}(t)\Big\{1+tT_{0,1,0}(t)^2T^{du}_{1,1,1}(t)\Big\},\\
T^{du}_{1,1,1}(t) &= 1+2tT^{du}_{1,1,1}(t)^2T^{ud}_{1,1,1}(t).
\end{aligned}$$

When $x=z=1, y=0$, by $T_{0,0,0}(t)=1$, (3.13) and (3.14) generate

$$\begin{aligned}
T^{ud}_{1,0,1}(t) &= 1-tT^{ud}_{1,0,1}(t)+2tT^{ud}_{1,0,1}(t)^2,\\
T^{du}_{1,0,1}(t) &= 1-tT^{du}_{1,0,1}(t)+2tT^{du}_{1,0,1}(t)^2T^{ud}_{1,0,1}(t),
\end{aligned}$$

from which, one can deduce that

$$\begin{aligned}
T^{ud}_{1,0,1}(t) &= R(t) = \frac{1+t-\sqrt{1-6t+t^2}}{4t} = \frac{1}{1+t}C(\frac{2t}{(1+t)^2}),\\
T^{du}_{1,0,1}(t) &= \frac{1+t-\sqrt{(1+t)^2-8tR(t)}}{4tR(t)}\\
&= \frac{1}{1+t}C(\frac{2tR(t)}{(1+t)^2}) = \sum_{i\geq 0}\frac{2^iC_iR(t)^it^i}{(1+t)^{2i+1}}\\
&= \sum_{i\geq 0}\frac{2^iC_it^i}{(1+t)^{3i+1}}\Big\{C(\frac{2t}{(1+t)^2})\Big\}^i\\
&= \sum_{i\geq 0}\frac{2^iC_it^i}{(1+t)^{3i+1}}\sum_{j\geq 0}\frac{i}{2j+i}\binom{2j+i}{j}\frac{2^jt^j}{(1+t)^{2j}}\\
&= \sum_{n\geq 0}t^n\sum_{i+j+k=n}(-1)^k\binom{3i+2j+k}{k}\frac{i}{2j+i}\binom{2j+i}{j}2^{i+j}C_i.
\end{aligned}$$

Hence we have

Theorem 3.5. *The set $\mathsf{GNC}_n(ud,h)$ of $\{ud,h\}$-avoiding GNC-trees is counted by the nth little Schröder number, and the set $\mathsf{GNC}_n(du,h)$ of $\{du,h\}$-avoiding GNC-trees is counted by*

$$|\mathsf{GNC}_n(du,h)| = \sum_{i+j+k=n}(-1)^k\binom{3i+2j+k}{k}\frac{i}{2j+i}\binom{2j+i}{j}2^{i+j}C_i.$$

This sequence beginning with $1,1,5,27,157,957,6025$ is not listed in [11].

3.3. **The pattern** $\{uu,dd\}$**.** Now we consider the pattern $\{uu,dd\}$, let $P=\{uu,dd\}$, according to the decomposition of GNC-trees, relations between $T^P_{x,y,z}(t)$ and $T^{*P}_{x,y,z}(t)$ can be derived, the details are omitted.

$$\begin{aligned}
T^P_{x,y,z}(t) &= 1+ytT_{0,y,0}(t)^2T^P_{x,y,z}(t)+ytT_{0,y,0}(t)\Big\{T^P_{x,y,z}(t)-T_{0,y,0}(t)\Big\}\Big\{2T^{*P}_{x,y,z}(t)-1\Big\}\\
&\quad + xt\Big\{2T^P_{z,y,x}(t)-T_{0,y,0}(t)\Big\}\Big\{1+ytT_{0,y,0}(t)^2T^P_{x,y,z}(t)\Big\}\Big\{2T^{*P}_{x,y,z}(t)-1\Big\},\\
T^{*P}_{x,y,z}(t) &= 1+xtT^P_{z,y,x}(t)\Big\{1+ytT_{0,y,0}(t)^2T^P_{x,y,z}(t)\Big\}\Big\{2T^{*P}_{x,y,z}(t)-1\Big\},
\end{aligned}$$

from which, one can get

$$T^P_{x,y,z}(t) = \Big\{1+ytT_{0,y,0}(t)^2T^P_{x,y,z}(t)\Big\}\Big\{1-xtT_{0,y,0}(t)^2+2xtT^P_{z,y,x}(t)T^P_{x,y,z}(t)\Big\}. \tag{3.15}$$

Let $x=y=z=1$ in (3.15), one has

Proposition 3.6. *The generating function for $\{uu,dd\}$-avoiding GNC-trees is given by*

$$T^P_{1,1,1}(t) = \Big\{1+tT_{0,1,0}(t)^2T^P_{1,1,1}(t)\Big\}\Big\{1-tT_{0,1,0}(t)^2+2tT^P_{1,1,1}(t)^2\Big\}.$$

Let $y=0$ in (3.15), by $T_{0,0,0}(t)=1$, one can get

$$T^P_{x,0,z}(t) = 1-xt+2xtT^P_{z,0,x}(t)T^P_{x,0,z}(t). \tag{3.16}$$

Exchanging x and z in (3.16), one has

$$T^P_{z,0,x}(t) = 1-zt+2ztT^P_{x,0,z}(t)T^P_{z,0,x}(t). \tag{3.17}$$

From (3.16) and (3.17), one can obtain

$$T^P_{x,0,z}(t) = 1-xt-2(z-x)tT^P_{x,0,z}(t)+2ztT^P_{x,0,z}(t)^2,$$

which leads to

$$T^P_{x,0,z}(t) = \frac{1+2(z-x)t-\sqrt{(1+2(z-x)t)^2-8zt(1-xt)}}{4zt}. \tag{3.18}$$

Setting $z = 1$ in (3.18), one can deduce

$$\begin{aligned} T^P_{x,0,1}(t) &= \frac{1+2(1-x)t-\sqrt{(1+2(1-x)t)^2-8t(1-xt)}}{4t} \\ &= \frac{1-xt}{1+2(1-x)t}C(\frac{2t(1-xt)}{(1+2(1-x)t)^2}) = \sum_{i\geq 0}\frac{2^iC_it^i(1-xt)^{i+1}}{(1+2(1-x)t)^{2i+1}} \\ &= \sum_{i\geq 0}2^iC_it^i\sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}x^jt^j\sum_{k\geq 0}(-1)^k\binom{2i+k}{k}2^kt^k\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}x^\ell \\ &= \sum_{n\geq 0}\sum_{r=0}^{n}t^nx^r\sum_{i+j+k=n}(-1)^{r+k}\binom{i+1}{j}\binom{2i+k}{k}\binom{k}{r-j}2^{i+k}C_i. \end{aligned}$$

Setting $x = z = 1$ in (3.18), one has

$$\begin{aligned} T^P_{1,0,1}(t) &= \frac{1-\sqrt{1-8t(1-t)}}{4t} = (1-t)C(2t(1-t)) \\ &= \sum_{n\geq 0}t^n\sum_{i=0}^{n}(-1)^{n-i}\binom{i+1}{n-i}2^iC_i. \end{aligned}$$

Hence we have

Theorem 3.7. *The number of $\{uu, dd, h\}$-avoiding (i.e., alternating) GNC-trees of $n+1$ points is given by*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{i+1}{n-i}2^iC_i, \qquad ([11, \text{A068764}]).$$

Precisely, the number of alternating GNC-trees of $n+1$ points with exactly r ascents is

$$\sum_{i+j+k=n}(-1)^{r+k}\binom{i+1}{j}\binom{2i+k}{k}\binom{k}{r-j}2^{i+k}C_i.$$

Remark 3.8. *When $r = 0$, then $j = 0$ and there has no alternating GNC-tree of $n+1$ points with exactly $r = 0$ ascents for $n \geq 1$, so we have*

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n+i}{n-i}C_i = 0, \qquad (n\geq 1),$$

which is a special case $q = 0$ of the Narayana polynomial identity [7]

$$\sum_{i=1}^{n}\frac{1}{n}\binom{n}{i-1}\binom{n}{i}q^i = \sum_{i=0}^{n}\binom{n+i}{n-i}\frac{1}{i+1}\binom{2i}{i}(q-1)^{n-i}.$$

When $x = -1, z = 1$ in (3.18), one has

$$\begin{aligned} T^P_{-1,0,1}(t) &= \frac{1+4t-\sqrt{1+8t^2}}{4t} = 1-tC(-2t^2) \\ &= 1+\sum_{n\geq 0}(-1)^{n+1}2^nC_nt^{2n+1}. \end{aligned}$$

Then we have

Theorem 3.9. *The parity of number of alternating GNC-trees of m points according to the even or odd number of ascents is zero if $m = 2n + 3$ and $(-1)^{n+1}2^n C_n$ if $m = 2n + 2$ for $n \geq 0$.*

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